Catastrophic Instability of Small Lovelock Black Holes

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Abstract

We study the stability of static black holes in Lovelock theory which is a natural higher dimensional generalization of Einstein theory. We show that Lovelock black holes are stable under vector perturbations in all dimensions. However, we prove that small Lovelock black holes are unstable under tensor perturbations in even-dimensions and under scalar perturbations in odd-dimensions. Therefore, we can conclude that small Lovelock black holes are unstable in any dimensions. The instability is stronger on small scales and hence catastrophic in the sense that there is no smooth descendant.

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I. INTRODUCTION

The possibility of higher dimensional black hole creation at the LHC would be the most fascinating prediction of the braneworld with large extra-dimensions [1]. Nowadays, it is fashionable to study higher dimensional black holes. The study of higher dimensional black holes is not a straightforward extension of the 4-dimensional one. In fact, in higher dimensions, there are many black objects with non-trivial topologies such as black rings. This seems to require the stability analysis of higher dimensional black objects. Moreover, in higher dimensions, Einstein theory is not the most general theory of gravity which contains terms only up to the second order derivatives in the equations of motion. The most general theory in this sense is Lovelock theory [2]. Thus, it is worth extending the stability analysis to this general Lovelock theory.

The stability of higher dimensional black holes has been intensively studied since the seminal papers by Kodama and Ishibashi [3]. It is important to study various black holes in Einstein theory because black holes produced at the LHC are expected to be charged or rotating. A numerical study of charged black holes has been done [4]. To investigate the stability of rotating black holes, a group theoretical method is developed [5]. The method is used to study the stability of squashed black holes [6–8] and 5-dimensional rotating black holes with equal angular momenta [9]. The stability of a special class of rotating black holes in more than 5-dimensions is also studied [10–12]. Recent development of numerical stability analysis is remarkable [13–17]. It has been shown that rapidly rotating black holes are unstable.

As we mentioned already, in higher dimensions, we need to extend Einstein theory to Lovelock theory. In addition to the theoretical requirement, we have a physical motivation to consider higher derivative theory of gravity. In fact, at the energy scale of black hole production, Einstein theory is not reliable any more. It is believed that string theory which can be consistently formulated only in 10-dimensions is the most promising candidate of the unified theory. We should recall that string theory predicts Einstein theory only in the low energy limit [18]. In string theory, there are higher curvature corrections in addition to the Einstein-Hilbert term [18]. Thus, it is natural to extend gravitational theory into those with higher power of curvature in higher dimensions. It is Lovelock theory that belongs to such class of theories [2, 19]. In Lovelock theory, it is known that there exist static

spherical symmetric black hole solutions [20] (and topological black hole solutions are also found in [21]). Hence, it is natural to suppose black holes produced at the LHC are of this type [22]. Thus, it is important to study the stability of these Lovelock black holes.

In the case of the second order Lovelock theory, the so-called Einstein-Gauss-Bonnet theory, the stability analysis under tensor perturbations has been performed [23] (see also an earlier work [24]). The analysis has been also extended to the scalar and vector perturbations [25]. It is shown that there exists the scalar mode instability in 5-dimensions, the tensor mode instability in 6-dimensions, and no instability in other dimensions. Although Einstein-Gauss-Bonnet theory is the most general theory in 5 and 6-dimensions, it is not so in more than 6-dimensions. For example, when we consider 10-dimensional black holes, we need to incorporate the fourth order Lovelock term. Indeed, when we consider black holes at the LHC, it is important to consider these higher order Lovelock terms [26]. Hence, the purpose of this paper is to study the stability of black holes in any order Lovelock theory, namely, in any dimensions. We have already shown that Lovelock black holes are unstable in even-dimensions under tensor perturbations [27]. In this paper, we extend previous results to vector and scalar perturbations using the master equations we have obtained recently [28].

The organization of this paper is as follows. In section II, we review Lovelock theory and explain a graphical method for constructing Lovelock black hole solutions. In section III, we consider tensor perturbations and show the instability of small Lovelock black holes in even-dimensions. This is a review of our previous paper [27]. In section IV, we show that black holes are stable under vector perturbations. In section V, we examine scalar perturbations and show that there exists the instability in odd-dimensions if black holes are sufficiently small. In section VI, we present a detailed analysis for Einstein-Gauss-Bonnet theory to illustrate our statements. The final section VII is devoted to the conclusion.

II.   LOVELOCK BLACK HOLES

In this section, we review Lovelock theory and introduce a graphical method to reveal the nature of asymptotically flat black hole solutions.

The most general divergence free symmetric tensor constructed out of a metric and its first and second derivatives has been obtained by Lovelock [2]. The corresponding Lagrangian

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can be constructed from m-th order Lovelock terms

$$\mathcal{L}_m = \frac{1}{2^m}\delta^{\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\rho_1\kappa_1\cdots\rho_m\kappa_m}R_{\lambda_1\sigma_1}{}^{\rho_1\kappa_1}\cdots R_{\lambda_m\sigma_m}{}^{\rho_m\kappa_m} \ , \tag{1}$$

where $R_{\lambda\sigma}{}^{\rho\kappa}$ is the Riemann tensor in D-dimensions and $\delta^{\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\rho_1\kappa_1\cdots\rho_m\kappa_m}$ is the generalized totally antisymmetric Kronecker delta. By construction, the Lovelock terms vanish for $2m > D$. It is also known that the Lovelock term with $2m = D$ is a topological term. Thus, Lovelock Lagrangian in D-dimensions is defined by

$$L = \sum_{m=0}^{k} c_m \mathcal{L}_m \ , \tag{2}$$

where we defined the maximum order $k \equiv [(D-1)/2]$ and c_m are arbitrary constants. Here, $[z]$ represents the maximum integer satisfying $[z] \leq z$. Hereafter, we set $c_0 = -2\Lambda$, $c_1 = 1$ and $c_m = a_m/m$ $(m \geq 2)$ for convenience. Taking variation of the Lagrangian with respect to the metric, we can derive Lovelock equation

$$0 = \mathcal{G}^\nu_\mu = \Lambda\delta^\nu_\mu - \sum_{m=1}^{k}\frac{1}{2^{(m+1)}}\frac{a_m}{m}\delta^{\nu\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\mu\rho_1\kappa_1\cdots\rho_m\kappa_m}R_{\lambda_1\sigma_1}{}^{\rho_1\kappa_1}\cdots R_{\lambda_m\sigma_m}{}^{\rho_m\kappa_m} \ . \tag{3}$$

As is shown in [20, 21], there exist static exact solutions of Lovelock equation. Let us consider the following metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\bar{\gamma}_{ij}dx^i dx^j \ , \tag{4}$$

where $\bar{\gamma}_{ij}$ is the metric of $n \equiv D-2$-dimensional constant curvature space with a curvature κ=1,0 or -1. Using this metric ansatz, we can calculate Riemann tensor components as

$$R_{tr}{}^{tr} = -\frac{f''}{2}, \ \ R_{ti}{}^{tj} = R_{ri}{}^{rj} = -\frac{f'}{2r}\delta^j_i, \ \ R_{ij}{}^{kl} = \left(\frac{\kappa-f}{r^2}\right)\left(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j\right) \ . \tag{5}$$

Substituting (5) into (3) and defining a new variable $\psi(r)$ by

$$f(r) = \kappa - r^2\psi(r) \ , \tag{6}$$

we obtain an algebraic equation

$$W[\psi] \equiv \sum_{m=2}^{k}\left[\frac{a_m}{m}\left\{\prod_{p=1}^{2m-2}(n-p)\right\}\psi^m\right] + \psi - \frac{2\Lambda}{n(n+1)} = \frac{\mu}{r^{n+1}} \ . \tag{7}$$

In (7), we used $n = D - 2$ and μ is a constant of integration which is related to the ADM mass as [29]:

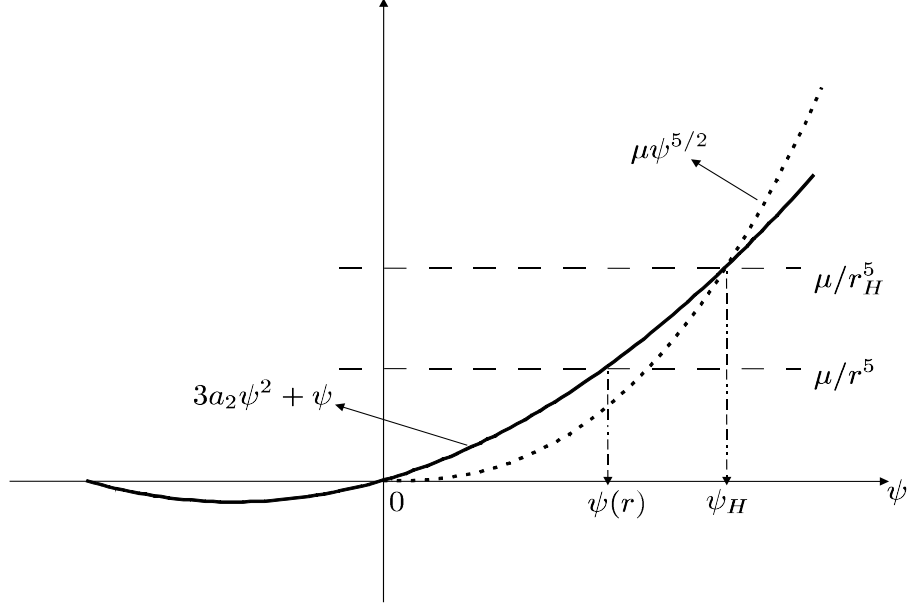$$M = \frac{2\mu\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \ , \tag{8}$$

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FIG. 1: The intersection between the solid curve and the thin horizontal line determines the solution $\psi = \psi(r)$ for the case $n = 4$. Apparently, the infinity $r = \infty$ corresponds to $\psi = 0$. The intersection between solid and dashed curve gives a horizon r_H.

where we used a unit $16\pi G = 1$.

From (7), it is easy to see that the solution $f(r)$ has many branches. In this paper, we want to concentrate on asymptotically flat spherically symmetric, i.e. $\Lambda = 0$ and $\kappa = 1$, solutions with a positive ADM mass $\mu > 0$ because such black holes could be created at the LHC. We also assume that Lovelock coefficients satisfy

$$a_m \geq 0 , \tag{9}$$

for simplicity. For example, in the case of $n = 3$, the theory is reduced to Einstein-Gauss-Bonnet theory. In this case, Eq.(7) reads

$$a_2 \psi^2 + \psi = \frac{\mu}{r^4} , \tag{10}$$

which can be explicitly solved as

$$\psi = \frac{-1 \pm \sqrt{1 + \frac{4a_2\mu}{r^4}}}{2a_2} . \tag{11}$$

The upper branch leads to the asymptotically flat solution

$$f(r) = 1 + \frac{r^2}{2a_2} \left[1 - \sqrt{1 + \frac{4a_2\mu}{r^4}} \right] . \tag{12}$$

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In the case of $n = 4$, Eq.(7) is reduced to

$$3a_2\psi^2 + \psi = \frac{\mu}{r^5} \ . \tag{13}$$

Although it is easy to solve this equation analytically, in Fig.1, a graphical method is also explained for this case. However, in the case of $n = 5$, Eq.(7) becomes the third order algebraic equation

$$8a_3\psi^3 + 6a_2\psi^2 + \psi = \frac{\mu}{r^6} \ . \tag{14}$$

We have a formula for solutions of Eq.(14), however, the roots are complicated in general. Hence, we use a graphical method illustrated in Fig.2. Because of the conditions (9), the function is monotonic for positive ψ. From (7), we see the root behaves $\psi \sim \mu/r^{n+1}$ or $f(r) \sim 1 - \mu/r^{n-1}$ as $r \to \infty$. Thus, the asymptotically flat solutions belong to the branch where ψ is always positive.
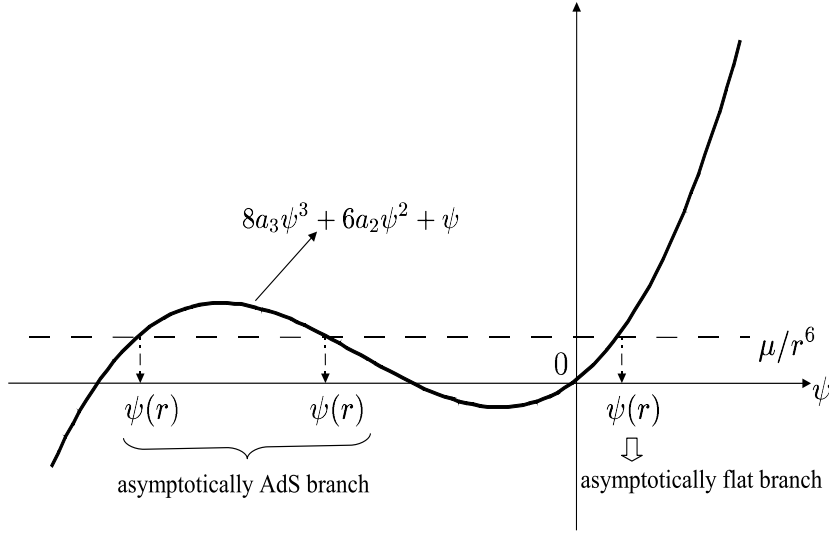

FIG. 2: We illustrate a graphical method for $n = 5$ case. In this case, the third order Lovelock theory is most general. Therefore, $W[\psi]$ in (7) reads a cubic polynomial. In this figure, three roots are depicted. Among these roots, only $\psi \geq 0$ one corresponds to an asymptotically flat solution.

Now, let us look for the horizon of black holes. The horizon radius of the asymptotically flat solution is characterized by $f(r_H) = 0$. From (6), we have a relation $\psi_H \equiv \psi(r_H) = 1/r_H^2$. Using this relation and (7), we obtain an algebraic equation

$$W[\psi_H] = \mu \psi_H^{(n+1)/2} \ . \tag{15}$$

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This determines ψ_H and hence r_H. For example, in the case of $n = 3$, we can easily solve this polynomial equation (15) as

$$\psi_H = \frac{1}{\mu - a_2} \ . \tag{16}$$

Note that if $\mu \leq a_2$, there appears a naked singularity. So, we consider the range $\mu > a_2$. In the case of n=4, since it is a bit complicated to solve Eq.(15) analytically, we present a graphically method in Fig.1. Similarly, in Fig.3, we present a graphical method to solve Eq.(15) for $n = 5$. From Fig.3, it is obvious that the range $\infty \geq r \geq r_H$ corresponds to $0 \leq \psi \leq \psi_H$ when $f(r)$ describes an asymptotically flat solution. It is also apparent that ψ_H becomes larger as μ becomes smaller.

Remarkably, the nature of black holes depends on the dimensions. In even-dimensions $n = 2k$, dividing (15) by ψ_H, we have

$$- \mu \psi_H^{(2k-1)/2} + \sum_{m=2}^{k} \left[\frac{a_m}{m} \left\{ \prod_{p=1}^{2m-2} (n-p) \right\} \psi_H^{m-1} \right] + 1 = 0 \ . \tag{17}$$

Near $\psi_H = 0$, the left hand side is positive; however, when ψ_H is sufficiently large, it is negative. Then, there exists a positive root somewhere between. This root moves from 0 to ∞ as μ moves from ∞ to 0. Thus, there is no restriction for μ in this case. On the other hand, in odd-dimensions $n = 2k - 1$, after dividing by ψ_H, Eq.(15) becomes

$$\left(\frac{a_k}{k} \left\{ \prod_{p=1}^{2k-2} (n-p) \right\} - \mu \right) \psi_H^{k-1} + \sum_{m=2}^{k-1} \left[\frac{a_m}{m} \left\{ \prod_{p=1}^{2m-2} (n-p) \right\} \psi_H^{m-1} \right] + 1 = 0 \ . \tag{18}$$

In order to have a positive root, we need $\mu > \frac{a_k}{k} \left\{ \prod_{p=1}^{2k-2} (n-p) \right\}$. Hence, we have the lower bound for the mass in odd-dimensions. In fact, there exists a positive root in this case, because the left hand side of Eq.(18) is positive near $\psi_H = 0$ and negative for sufficiently large ψ_H. Furthermore, It is not difficult to see that this root approaches 0 as $\mu \to \infty$ and approaches ∞ as $\mu \to \frac{a_k}{k} \left\{ \prod_{p=1}^{2k-2} (n-p) \right\}$. Therefore, the root ψ_H moves in the range $0 < \psi_H < \infty$ as μ moves in the range $\frac{a_k}{k} \left\{ \prod_{p=1}^{2k-2} (n-p) \right\} < \mu < \infty$.

Finally, we examine the singularity in the solutions. Using the metric ansatz (4), the Kretschmann scalar $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$ is

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = f'' + 2n \frac{f'^2}{r^2} + 2n(n-1) \frac{(\kappa - f)^2}{r^4} \ .$$
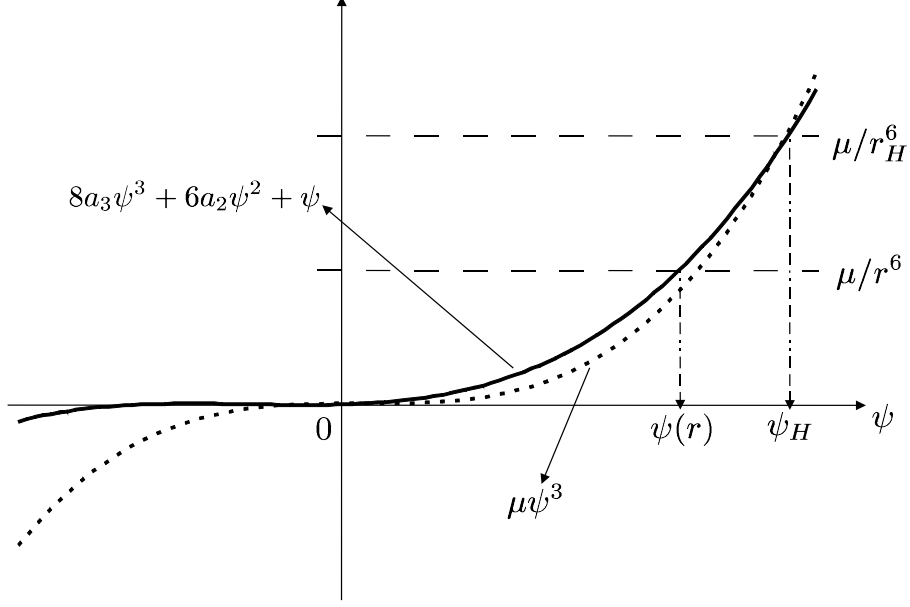
FIG. 3: For $n = 5$ case, this figure explains a method for calculating ψ_H or r_H graphically. The positive root of Eq.(15) ψ_H can be obtained from the intersection of the solid and dashed curves. Since the intersection between the horizontal line and the solid curve gives a solution $\psi = \psi(r)$, the horizon radius r_H is determined from the intersection of the horizontal line, the solid and dashed curves.

Thus, this solution has curvature singularities at $r = 0$ or at the point where derivatives of $f(r)$ diverges. For example, f' diverges at the point where ψ' diverges because of $f' = -2r\psi + r^2\psi'$. Taking a derivative of (7) with respect to r, we have a relation

$$\psi' = -\frac{(n+1)\mu}{r^{n+2}\partial_\psi W[\psi]} \ .$$

Hence, a curvature singularity appears at the point $\partial_\psi W = 0$. However, for the asymptotically flat branch, $W[\psi]$ is monotonically increasing function of ψ. Hence, we conclude $\partial_\psi W[\psi] > 0$. Similarly, for asymptotically flat spherical symmetric solutions, we can prove that the conditions $|1 - f| < \infty$, $|f'| < \infty$ and $|f''| < \infty$ are satisfied except for $r = 0$. Therefore, there is a curvature singularity only at $r = 0$ in the cases we are considering. And asymptotically flat solutions have the horizon if the parameter μ is sufficiently large. So, these solutions do not have a naked singularity and describe black holes with a mass M defined by Eq.(8).

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III. STABILITY ANALYSIS FOR TENSOR PERTURBATIONS

In this section, we examine the stability under tensor perturbations, which we have already studied in [27].

We start from the master equation for tensor perturbations

$$- f^2 \chi'' - \left(f^2 \frac{T''}{T'} + \frac{2f^2}{r} + f f' \right) \chi' + \frac{(2\kappa + \gamma_t) f}{(n-2)r} \frac{T''}{T'} \chi = \omega^2 \chi \ , \tag{19}$$

where ω is a frequency and we have defined a key function

$$T(r) = r^{n-1} \partial_\psi W[\psi] \ . \tag{20}$$

In (19), χ is the master variable and γ_t is eigenvalue of tensor harmonics which is given by $\gamma_t = \ell(\ell + n - 1) - 2$, $(\ell = 2, 3, 4 \cdots)$ for $\kappa = 1$ and positive real numbers for $\kappa = -1, 0$.

Here, we should recall our assumptions. We assumed the conditions (9) are satisfied. And we also assumed spherical symmetry and positivity of the mass, i.e., $\kappa = 1$ and $\mu > 0$. Then, there exists an asymptotically flat spherical symmetric branch which we have considered in section II. Note that $T(r)$ which is defined above is always positive in this branch.

A. Criterion for Stability

We will present the condition for the stability of the solutions we are considering.

As we will soon see, the master equation (19) can be transformed into the Schrödinger form. To do this, we have to impose the condition

$$T'(r) > 0 \ , \quad (\text{for } r > r_H) \ . \tag{21}$$

In fact, this is necessary for the linear analysis to be applicable. In the case that there exists r_0 such that $T'(r_0) = 0$ and $r_0 > r_H$, we encounter a singularity. Using approximations $T'(r) \sim T''(r_0)(r - r_0) \equiv T''(r_0)y$, $f(r) = f(r_0)$ and $r = r_0$, (19) approximately becomes

$$y \frac{d^2 \chi}{dy^2} + \frac{d\chi}{dy} + c\chi = 0 \ . \tag{22}$$

This shows that near $r = r_0$, χ behaves as $\chi \sim c_1 + c_2 \log y$, where c_1 and c_2 are constants of integration. Hence, the solution is singular at $y = 0$ for generic perturbations. The similar situation occurs even in cosmology with higher derivative terms [30, 31]. In those cases,

this kind of singularity alludes to ghosts. Indeed, if there is a region $T'(r) < 0$ outside the horizon, the kinetic term of perturbations has a wrong sign. Hereafter, we call this the ghost instability.

When the condition (21) is fulfilled, introducing a new variable $\Psi(r) = \chi(r)r\sqrt{T'(r)}$ and switching to the coordinate r^*, defined by $dr^*/dr = 1/f$, we can rewrite Eq.(19) as

$$-\frac{d^2\Psi}{dr^{*2}} + V_t(r(r^*))\Psi = \omega^2\Psi \ , \tag{23}$$

where

$$V_t(r) = \frac{(2\kappa + \gamma_t)f}{(n-2)r}\frac{d\ln T'}{dr} + \frac{1}{r\sqrt{T'}}f\frac{d}{dr}\left(f\frac{d}{dr}r\sqrt{T'}\right) \tag{24}$$

is an effective potential.

For discussing the stability, the "S-deformation" approach is useful [3, 23]. Let us define the operator

$$\mathcal{H} \equiv -\frac{d^2}{dr^{*2}} + V_t \tag{25}$$

acting on smooth functions defined on $I = (r_H^*, \infty)$. Then, (23) is the eigenequation and ω^2 is eigenvalue of $\mathcal{H}$. We also define the inner products as

$$(\varphi_1, \varphi_2) = \int_I \varphi_1^* \varphi_2 dr^* \ . \tag{26}$$

In this case, for any φ, we can find a smooth function S such that

$$(\varphi, \mathcal{H}\varphi) = \int_I (|D\varphi|^2 + \tilde{V}|\varphi|^2)dr^*, \tag{27}$$

where we have defined

$$D = \frac{d}{dr^*} + S \ , \quad \tilde{V} = V_t + f\frac{dS}{dr} - S^2 \ . \tag{28}$$

Following [23], we choose S to be

$$S = -f\frac{d}{dr}\ln\left(r\sqrt{T'}\right) \ . \tag{29}$$

Then, we obtain the formula

$$(\varphi, \mathcal{H}\varphi) = \int_I |D\varphi|^2 dr^* + (2\kappa + \gamma_t)\int_{r_H}^{\infty} \frac{|\varphi|^2}{(n-2)r}\frac{d\ln T'}{dr}dr \ . \tag{30}$$

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Here, the point is that the second term in (30) includes a factor $2\kappa + \gamma_t > 0$, but T' does not include γ_t. Hence, by taking a sufficiently large $2\kappa + \gamma_t$, we can always make the second term dominant.

Now, let us show that the sign of $d \ln T'/dr$ determines the stability. If $d \ln T'/dr > 0$ on I, the solution (4) is stable. This can be understood as follows. Note that $2\kappa + \gamma_t > 0$, then we have $\tilde{V} > 0$ for this case. That means $(\varphi, \mathcal{H}\varphi) > 0$ for arbitrary φ if $d \ln T'/dr > 0$ on I. We choose, for example, φ as the lowest eigenstate, then we can conclude that the lowest eigenvalue ω_0^2 is positive. Thus, we proved the stability. The other way around, if $d \ln T'/dr < 0$ at some point in I, the solution is unstable. To prove this, the inequality

$$\frac{(\varphi, \mathcal{H}\varphi)}{(\varphi, \varphi)} \geq \omega_0^2 \tag{31}$$

is useful. This inequality is correct for arbitrary φ. If $d \ln T'/dr < 0$ at some point in I, we can find φ such that

$$\int_{r_H}^{\infty} \frac{|\varphi|^2}{(n-2)r} \frac{d \ln T'}{dr} dr < 0 \ . \tag{32}$$

In this case, (30) is negative for sufficiently large $2\kappa + \gamma_t$. Then, the inequality (31) implies $\omega_0^2 < 0$ and the solution has unstable modes. Thus, we can conclude that the solution is stable if and only if $d \ln T'/dr > 0$ on I.

From the above logic, if $d \ln T'/dr$ has a negative region, negative ω^2 states exist. Therefore, this instability is dynamical. Then, we call this as dynamical instability in order to distinguish this from the ghost instability which is caused by negativity of $T'(r)$.

We want to summarize this subsection. If T' has negative region outside the horizon $r > r_H$, Lovelock black holes have the ghost instability. Even if T' is always positive, Lovelock black holes have the dynamical instability if T'' has a negative region outside the horizon. Therefore, Lovelock black holes are stable under tensor perturbations if and only if T' and T'' are always positive outside the horizon.

B. Instability of Small Lovelock Black Holes in Even-dimensions

In this subsection, we check the sign of T' and T'' for asymptotic flat solutions $f(r)$. In order to see the sign of these functions, it is useful to express them as functions of ψ instead

of r. Using Eq.(7) and its derivative, we obtain

$$\left(\partial_\psi W[\psi]\right) \psi' = -(n+1)\frac{\mu}{r^{n+2}} = -(n+1)\frac{W[\psi]}{r} \ . \tag{33}$$

The above formula can be used to eliminate ψ' in T'. The result reads

$$T'(r) = \frac{r^{n-2}}{\partial_\psi W}\left[(n-1)\left(\partial_\psi W\right)^2 - (n+1)W\partial_\psi^2 W\right] \ . \tag{34}$$

Similarly, T'' can be written as

$$T'' = \frac{r^{n-3}}{(\partial_\psi W)^3}\left[(n-1)(n-2)\left(\partial_\psi W\right)^4 - (n+1)(n-4)W\left(\partial_\psi W\right)^2\partial_\psi^2 W\right.$$
$$\left. +(n+1)^2 W^2\left\{\partial_\psi W\partial_\psi^3 W - (\partial_\psi^2 W)^2\right\}\right] \ . \tag{35}$$

Since $W[\psi]$ is a polynomial function of ψ and $\partial_\psi W$ is positive, we can determine the sign of T' and T'' by examining the sign of polynomial in the numerators. Thus, the stability problem has been reduced to an algebraic one.

Substituting the general form of $W[\psi]$ into T', we obtain

$$T'(r) = r^{n-2}\frac{K[\psi]}{1 + \sum_{m=2}^{k}\left[a_m\left\{\prod_{p=1}^{2m-2}(n-p)\right\}\psi^m\right]} \ , \tag{36}$$

where

$$K[\psi] = (n-1) + \sum_{m=2}^{k}\left[a_m(n-1)\left\{(3-m)n - (m+1)\right\}\left\{\prod_{p=2}^{2m-2}(n-p)\right\}\psi^{m-1}\right]$$
$$+ \sum_{m,j=2}^{k}\left[a_m a_j(n-1)\left\{\prod_{p=2}^{2m-2}(n-p)\right\}\left\{\prod_{p=1}^{2j-2}(n-p)\right\}\right.$$
$$\left. \times\frac{j(n-1) - (m-1)(n+1)}{j}\psi^{j+m-2}\right] \ . \tag{37}$$

The factor other than $K[\psi]$ in (36) are manifestly positive, so the sign of $K[\psi]$ determines that of T'. However, from (37), it is clear that sign of $K[\psi]$ depends on dimensions and Lovelock coefficients a_m. Therefore, if Lovelock black holes have the ghost instability depends on dimensions and Lovelock coefficients.

However, as we will show later, T'' has a negative region in even-dimensions if Lovelock black holes are sufficiently small. Therefore, even if T' are always positive and consequently

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Lovelock black holes have no ghost instability, they have the dynamical instability as long as the ADM mass is sufficiently small.

Substituting the explicit form of $W[\psi]$ into the formula (35), we get

$$T'' = r^{n-3} \frac{L[\psi]}{\left(1 + \sum_{m=2}^{k} \left[a_m \left\{\prod_{p=1}^{2m-2}(n-p)\right\}\psi^m\right]\right)^3} \cdot \tag{38}$$

Here, the lowest and the leading term of $L[\psi]$ is

$$
\begin{aligned}
L[\psi] &= (n-1)(n-2) + \cdots \\
&\quad + \frac{a_k^4}{k^2}\left\{\prod_{p=1}^{2k-2}(n-p)^4\right\}(n-(2k-1))(n-(3k-1))\psi^{4k-4} \ .
\end{aligned}
\tag{39}
$$

We note that the highest order $k = [(D-1)/2]$ is related to dimensions as $n = 2k-1$ in odd-dimensions and $n = 2k$ in even-dimensions. In odd-dimensions, the leading term disappears. Hence, we cannot say anything in general. Hence, we consider only even-dimensions.

Let us examine the sign of $L[\psi]$ ($\psi \geq 0$). If $n = 2k$, the coefficient of the lowest term is positive and that of the leading one of (39) is negative. Therefore, $L[\psi] > 0$ near $\psi = 0$ and $L[\psi] < 0$ for large ψ. This means that there exists roots of $L[\psi] = 0$ because $L[\psi]$ is a continuous function. Let ψ_0 be the lowest positive root. If $\psi_H < \psi_0$, then $L[\psi] > 0$ for $0 \leq \psi \leq \psi_H$, and hence we conclude $T'' > 0$ for $r > r_H$. While, if $\psi_H > \psi_0$, then there exists a region $L[\psi] < 0$ in the range $\psi_0 \leq \psi \leq \psi_H$. Thus, there exists a region $T'' < 0$ outside the horizon $r > r_H$. Therefore, black holes are stable if $\psi_H < \psi_0$ and unstable if $\psi_H > \psi_0$. Since ψ_H becomes larger as μ becomes smaller, we conclude that there exist a critical mass below which black holes become unstable.

To conclude this section, by considering tensor perturbations, we can say that "When the ADM mass is sufficiently small, Lovelock black holes in even-dimensions have the ghost instability or the dynamical instability; that is, small Lovelock black holes are unstable in even-dimensions ".

IV. STABILITY ANALYSIS FOR VECTOR PERTURBATIONS

In this section, we consider the stability of Lovelock black holes under vector perturbations.

Master equation for vector perturbation is given by [28];

$$-\partial_{r*}^2 \Psi + V_v(r)\Psi = \omega^2 \Psi \tag{40}$$

where

$$V_v(r) = \frac{1}{\frac{1}{r\sqrt{T'}}} f\partial_r \left(f\partial_r \frac{1}{r\sqrt{T'}} \right) + \left(\frac{\gamma_v}{n-1} - \kappa \right) \frac{fT'}{rT} . \tag{41}$$

Here, Ψ is the master variable and γ_v are eigenvalues of vector harmonics with $\gamma_v = \ell(\ell + n - 1) - 1$ ($\ell \geq 1$) for $\kappa = 1$ and non-negative real numbers for $\kappa = 0, -1$. This equation is obtained provided that tensor perturbations have no ghost instability. Moreover, $T(r)$ is always positive, because we assumed the conditions (9), the positivity of μ , and the asymptotically flat spherical symmetric branch.

In this section, we show that Lovelock black holes are stable under vector perturbations as long as they have no ghost instability under tensor perturbations. In order to prove this statement, we again use the S-deformation approach. We define $\mathcal{H} = -d^2/dr^{*2} + V_v(r)$ and the inner product (26). Then, as in the last section, we can find smooth function S such that

$$(\varphi, \mathcal{H}\varphi) = \int_I (|D\varphi|^2 + \tilde{V}|\varphi|^2)dr^*,$$

for any φ . Here, we have defined

$$D = \frac{d}{dr^*} + S , \quad \tilde{V} = V_v + f\frac{dS}{dr} - S^2 .$$

Following the paper [25], we choose S to be

$$S = -f\frac{d}{dr}\ln(\frac{1}{r\sqrt{T'}}) . \tag{42}$$

Then, $\tilde{V}$ can be calculated as

$$\tilde{V} = \frac{\gamma_v - (n-1)\kappa}{n-1} \frac{fT'}{rT} . \tag{43}$$

In this new potential $\tilde{V}$, apparently $\gamma_v - (n-1)\kappa > 0$, $f > 0$ and $T > 0$. We also assumed $T' > 0$, so it is clear that $\tilde{V}$ is always positive. Therefore, the inner product $(\varphi, \mathcal{H}\varphi)$ is positive for any φ. In particular, it is true for the lowest energy state, hence the lowest energy is positive. This implies that black holes are stable under vector perturbations.

To summarize this section, we can say that " if there is no ghost instability in tensor perturbation, Lovelock black holes are stable under vector perturbations".

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V. STABILITY ANALYSIS FOR SCALAR PERTURBATIONS

In this section, we examine the stability of Lovelock black holes under scalar perturbations.

In the previous paper, we have derived the master equation for scalar perturbations [28]. Using the master variable Ψ, we can write down the master equation

$$- \partial_{r^*}^2 \Psi + V_s(r)\Psi = \omega^2 \Psi \ . \tag{44}$$

Here, the effective potential reads

$$
\begin{aligned}
V_s(r) &= 2\gamma_s f \frac{(rNT)'}{nNTr^2} \\
&\quad - \frac{f}{N}\partial_r\left(f\partial_r N\right) + 2f^2\frac{N'^2}{N^2} - \frac{f}{T}\partial_r(f\partial_r T) + 2f^2\frac{T'^2}{T^2} + 2f^2\frac{N'T'}{NT} \ ,
\end{aligned}
\tag{45}
$$

where we have defined

$$N(r) = \frac{-2nf + 2\gamma_s + nrf'}{r\sqrt{T'}} \ . \tag{46}$$

For scalar perturbations, eigenvalues of scalar harmonics γ_s are given by $\gamma_s = \ell(\ell + n - 1)$ for $\kappa = 1$ and positive real numbers for $\kappa = 0, -1$. The above master equation is obtained by assuming $T' > 0$. Hence, tensor perturbations have no ghost instability.

Note that we will consider an asymptotically flat spherically symmetric branch with positive mass as in section II. In this branch, $T(r)$ is always positive.

A. Criterion for Instability

In this subsection, we show that black holes are unstable if $2T'^2 - TT''$ has a negative region outside the horizon.

In order to prove this statement, we can use the S-deformation approach. Here, we choose S as

$$S = f\partial_r(\ln N) + f\partial_r(\ln T) \ , \tag{47}$$

then the second line of (45) canceled and $\tilde{V}$ becomes

$$
\begin{aligned}
\tilde{V} &= 2\gamma_s f \frac{(rNT)'}{nNTr^2} \\
&= \frac{2\gamma_s f}{nr} \left[\frac{2(\gamma_s - n\kappa)}{2(\gamma_s - n\kappa) + \frac{n(n+1)\mu}{T}} \frac{T'}{T} - \frac{1}{2}\frac{T''}{T'} \right] \\
&< \frac{\gamma_s f}{nrTT'} \left[2T'^2 - TT'' \right] .
\end{aligned}
\tag{48}
$$

Note that we used the assumption $T > 0$, $T' > 0$ and $\mu > 0$ in the last inequality.

Now, let us prove the statement "if $2T'^2 - TT''$ has a negative region, black holes are unstable". In order to do that, we use the inequality

$$
\omega_0^2 \le \frac{(\varphi, \mathcal{H}\varphi)}{(\varphi, \varphi)} ,
\tag{49}
$$

where ω_0^2 is the lowest eigenvalue. This inequality is true for arbitrary test function φ, so we choose φ as the smooth function that has compact support in the region where $2T'^2 - TT''$ is negative. Then, ω_0^2 can be bounded as

$$
\begin{aligned}
\omega_0^2 &\le (\varphi, \mathcal{H}\varphi)/(\varphi, \varphi) \\
&= \frac{1}{(\varphi, \varphi)} \int dr^* \left[|D\varphi|^2 + \tilde{V} |\varphi|^2 \right] \\
&< \frac{1}{(\varphi, \varphi)} \left[\int dr^* |D\varphi|^2 + \gamma_s \int dr^* \frac{f}{nrTT'} \left(2T'^2 - TT'' \right) |\varphi|^2 \right] .
\end{aligned}
\tag{50}
$$

We assume $T > 0$, $T' > 0$ and choose φ as the smooth function which has compact support in the region $2T'^2 - TT'' < 0$, so the first term in (50) must be positive and the second term in (50) must be negative. Therefore, by taking the limit $\gamma_s = \ell(\ell + n - 1) \to \infty$, the last line of Eq.(50) becomes negative, which means the lowest eigenvalue ω_0^2 is negative. Hence, black holes are dynamically unstable.

In summary, we can say that "If T' has a negative region, black holes have the ghost instability under tensor perturbations. Even if T' is always positive, Lovelock black holes have the dynamical instability under scalar perturbations if $2T'^2 - TT''$ has a negative region". Note that we can not say that black holes are stable even if $2T'^2 - TT''$ is always positive.

B. Instability of Small Lovelock Black Holes in Odd-dimensions

Now let us check the sign of T' and $2T'^2 - TT''$. We have already shown that Lovelock black holes are unstable in even-dimensions under tensor perturbations, so we concentrate on odd-dimensions. In order to examine the sign of these functions, it is convenient to express these functions by ψ. The formula (34) reads

$$T'(r) = r^{n-2} \frac{K[\psi]}{\partial_\psi W[\psi]} ,\qquad (51)$$

where

$$K[\psi] = (n-1)(\partial_\psi W)^2 - (n+1)W\partial_\psi^2 W .\qquad (52)$$

Similarly, the formula (35) can be written as

$$2T'^2 - TT'' = \frac{r^{2n-4}}{(\partial_\psi W)^2} M[\psi] ,\qquad (53)$$

where

$$M[\psi] = n(n-1)(\partial_\psi W)^4 - 3n(n+1)W\partial_\psi^2 W(\partial_\psi W)^2$$
$$+3(n+1)^2 W^2(\partial_\psi^2 W)^2 - (n+1)^2 W^2 \partial_\psi W \partial_\psi^3 W .\qquad (54)$$

Apparently, the signs of T' and $2T'^2 - TT''$ are determined by $K[\psi]$ and $M[\psi]$. So we have to check the signs of $K[\psi]$ and $M[\psi]$.

First, we consider the sign of T' which has been already expressed by ψ in (36) and (37). We here consider odd-dimensions $n = 2k - 1$, then we have

$$K[\psi] = 2(k-1) + \cdots + 2\frac{\alpha_{k-1}^2(k-2) - 2\alpha_k\alpha_{k-2}(k-1)}{(k-1)(k-2)}\psi^{2k-4} .\qquad (55)$$

where $\alpha_m = a_m \left\{ \prod_{p=1}^{2m-2}(n-p) \right\}$. Furthermore, for $n = 2k - 1$, $M[\psi]$ can be expressed by ψ as

$$M[\psi] = 2(2k-1)(k-1) + \cdots - 6a_k^2\frac{\alpha_{k-1}^2(k-2) - 2\alpha_k\alpha_{k-2}(k-1)}{(k-1)(k-2)}\psi^{4k-6} .\qquad (56)$$

The most important point is that the signs of the coefficients of the leading term in (55) and (56) are opposite.

First, we assume $\alpha_{k-1}^2(k-2) - 2\alpha_k\alpha_{k-2}(k-1)$ is negative. In this case, $K[\psi]$ is positive near $\psi = 0$ and negative for large ψ, so $K[\psi]$ has positive roots. Let the lowest root be

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ψ_0. Note that ψ moves in the range $(0, \psi_H]$. Then, if $\psi_H > \psi_0$, $K[\psi]$ has a negative region and T' does so, which means Lovelock black holes have the ghost instability. Note that ψ_H becomes larger as μ becomes smaller. Then, if $a_{k-1}^2(k-2) - 2a_k a_{k-2}(k-1)$ is negative, small Lovelock black holes have the ghost instability.

Conversely, we assume $\alpha_{k-1}^2(k-2) - 2\alpha_k \alpha_{k-2}(k-1)$ is positive. In this case, the coefficient of the leading term of $K[\psi]$ is positive, so whether $K[\psi]$ has positive root or not is obscure. If this has a positive root, we can say that small Lovelock black holes have the ghost instability by using the same logic as the last paragraph. However, even if $K[\psi]$ has no positive root, $M[\psi]$ must have positive roots; this is because $M[\psi]$ becomes positive near $\psi = 0$ and must be negative for large ψ. Let the lowest positive root be ψ_1. Using this definition, we can say Lovelock black holes have the dynamical instability if $\psi_H > \psi_1$, namely, if Lovelock black holes are sufficiently small.

In any case, assuming black holes are sufficiently small, Lovelock black hole have the ghost instability or the dynamical instability in odd-dimensions.

To conclude this section, we can say that "small Lovelock black holes are unstable in odd-dimensions".

VI.   EXAMPLE: EINSTEIN-GAUSS-BONNET THEORY

In this section, to illustrate our general statement, we take Einstein-Gauss-Bonnet theory as an example. The Einstein-Gauss-Bonnet theory corresponds to $k = 2$ and generic for $n = 3$ and $n = 4$.

The action for Einstein-Gauss-Bonnet theory is given by

$$L = R + \frac{a_2}{2} \left(R^2 - 4R_\mu{}^\nu R_\nu{}^\mu + R_{\mu\nu}{}^{\lambda\rho} R_{\lambda\rho}{}^{\mu\nu} \right) \ . \tag{57}$$

The stability of this model has been already analyzed in [23] and [25]. However, the derivation presented here is more explicit and hence transparent.

Substituting the concrete form of $W[\psi]$ into T', we can get

$$K[\psi] = (n-1) + 2a_2(n-1)(n-2)(n-3)\psi + a_2^2(n-1)^2(n-2)^2(n-3)\psi^2 \ . \tag{58}$$

As we can see from (58), T' is always positive both for $n = 3$ and $n = 4$ because we are considering cases $\psi > 0$ and $a_2 > 0$. Similarly, substituting the explicit form of $W[\psi]$ into

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T'', we obtain

$$L[\psi] = (n-1)^4(n-2)^4(n-3)(n-5)a_2^4\psi^4 + 4(n-1)^3(n-2)^3(n-3)(n-5)a_2^3\psi^3$$

$$+2(n-1)^2(n-2)^2(5n^2 - 25n + 42)a_2^2\psi^2$$

$$+12(n-1)(n-2)(n^2 - 3n + 4)a_2\psi + 4(n-1)(n-2) . \tag{59}$$

Finally, $M[\psi]$ which determines the sign of $2T'^2 - TT''$ can be calculated as

$$M[\psi] = \frac{1}{4}(n-1)^3(n-2)^4(n-3)a_2^4\psi^4 + (n-1)^4(n-2)^3(n-3)a_2^3\psi^3$$

$$+\frac{3}{2}(n-1)^2(n-2)^2(n^2 - 5n + 2)a_2^2\psi^2$$

$$+n(n-1)(n-2)(n-7)a_2\psi + n(n-1) . \tag{60}$$

The functions $L[\psi]$ and $M[\psi]$ give criterions for the instability of tensor and scalar perturbations, respectively.

Let us first consider 6-dimensional Einstein-Gauss-Bonnet black holes corresponding to $n = 4$. Then, substituting $n = 4$ into (60), we obtain

$$M[\psi] = 12(-1 + 3a_2\psi + 9a_2^2\psi^2)^2 > 0 . \tag{61}$$

Therefore, black holes are stable under scalar perturbations. Next, we need to check the stability of Lovelock black holes under tensor perturbations. In the case of $n = 4$, the formula (59) becomes

$$L[\psi] = -1296a_2^4\psi^4 - 864a_2^3\psi^3 + 1584a_2^2\psi^2 + 576a_2\psi + 24 . \tag{62}$$

Clearly, the coefficient of the leading term of $L[\psi]$ is negative. Thus, for large ψ, $L[\psi]$ is negative. And, $L[\psi]$ is positive near $\psi \sim 0$ because $L[0] = 24 > 0$. Hence, there must be a root somewhere between. Indeed, $L[\psi]$ becomes zero at

$$\psi_0 = \frac{1}{6a_2}(-1 + \sqrt{15} + \sqrt{10}) . \tag{63}$$

Therefore, $L[\psi]$ is always positive in the range $0 < \psi < \psi_0$ and always negative in the range $\psi > \psi_0$. As we explained, ψ moves in the range $0 < \psi \leq \psi_H$. Therefore, if $\psi_H > \psi_0$, there exists the region $T'' < 0$, which means Lovelock black holes are dynamically unstable. Furthermore, from (15), the inequality $\psi_H > \psi_0$ yields

$$\mu < \frac{3\sqrt{6}\left(1 + \sqrt{15} + \sqrt{10}\right)}{\left(-1 + \sqrt{15} + \sqrt{10}\right)^{3/2}}a_2^{3/2} \equiv \mu_c . \tag{64}$$

This proves that Lovelock black holes with the mass less than μ_c are unstable in 6 dimensions.

Now, let us consider 5-dimensional Einstein-Gauss-Bonnet black holes corresponding to $n = 3$. In the case of $n = 3$, the formula (59) reads

$$L[\psi] = 96a_2\psi^2 + 96a_2\psi + 8 . \tag{65}$$

Since all the coefficient of $L[\psi]$ are positive and we are considering positive ψ, $L[\psi]$ is always positive and hence T'' is always positive. Thus, Lovelock black holes in 5-dimensions are stable under tensor perturbations. However, for $n = 3$, $M[\psi]$ becomes

$$M[\psi] = 6(1 - 4a_2\psi - 4a_2^2\psi^2) . \tag{66}$$

From this equation, it is easy to see that $M[\psi] = 0$ has a positive solution; that is $\psi_0 = \frac{\sqrt{2}-1}{2a_2}$. Then, $M[\psi]$ is positive in the range $0 < \psi < \psi_0$ and negative in the range $\psi > \psi_0$. Since ψ moves in the range $0 < \psi \leq \psi_H$, $2T'^2 - TT''$ has a negative region if $\psi_H > \psi_0$. Using the solution for ψ_H (16), we can rewrite the inequality $\psi_H > \psi_0$ as

$$a_2 < \mu < (\sqrt{2} + 1)^2 a_2 . \tag{67}$$

Note that the lower bound came from the condition for the existence of the horizon as we have explained in Sec.II. Hence, 5-dimensional Lovelock black holes with the mass in the above range are dynamically unstable under scalar perturbations.

VII. CONCLUSION

We have studied the stability of static black holes in Lovelock theory which is a natural higher dimensional generalization of Einstein theory. We have shown that there exists the instability of Lovelock black holes with small mass under tensor perturbations in even-dimensions and under scalar perturbations in odd-dimensions. Lovelock black holes are stable under vector perturbations as long as they do not have ghost instability under tensor perturbations. Remarkably, the instability is stronger on short distance scales, which is different from the usual instability for which the onset of the instability becomes a bifurcation point. Hence, the instability we have discussed in this paper is catastrophic in the sense that there is no smooth descendant. Curiously, the similar instability also appears in the Gauss-Bonnet cosmology [32]. In spite of this unusual nature of the instability, it is interesting

to investigate the fate of the catastrophic instability. This issue is very important because black holes lose their mass due to the Hawking radiation and eventually become unstable.

It is worth examining the more profound meaning of this instability; that is, why Lovelock black holes are unstable. Especially, it is very interesting to find the reason why black holes have the instability under tensor perturbations in even-dimensions and under scalar perturbations in odd-dimensions.

Related to the above, it is intriguing to find the thermodynamical meaning of the universal function $T(r)$. As was shown in this paper, this function governs the dynamical stability of black holes. Therefore, if $T(r)$ has thermodynamical meaning, the relation between thermodynamical [21, 33] and dynamical instability might be revealed.

It is interesting to investigate if the instability we found also exists for asymptotically AdS cases from the point of view of the AdS/CFT correspondence [34–36], in particular, in relation to stability of holographic superconductors [37, 38].

Acknowledgments

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